Abstract

To determine whether or not a given graph has a hamiltonian cycle is much harder than deciding whether it is Eulerian, and no algorithmically useful characterization of hamiltonian graphs is known, although several necessary conditions and many sufficient conditions (see [6]) have been discovered. In fact, it is known that determining whether there are hamiltonian paths or cycles in arbitrary graphs is $\mathcal{NP}$-complete. The interested reader is referred in particular to the surveys of Berge ([5], Chapter 10), Bondy and Murty ([10], Chapters 4 and 9), J. C. Bermond [6], Flandrin, Faudree and Ryj*áč*stek [21] and R. Gould [27].

Hamiltonicity in special classes of graphs is a major area of graph theory and a lot of graph theorists have studied it. One special class of graphs whose hamiltonicity has been studied is that of Toeplitz graphs, introduced by van Dal et al. [13] in 1996. This study was continued by C. Heuberger [32] in 2002. The Toeplitz graphs investigated in [13] and [32] were all undirected. We intend to extend here this study to the directed case.

A *Toeplitz matrix*, named after Otto Toeplitz, is a square matrix ($n \times n$) which has constant values along all diagonals parallel to the main diagonal. Thus, Toeplitz matrices are defined by $2n-1$ numbers. Toeplitz matrices have uses in different areas in pure and applied mathematics, and also in computer science. For example, they are closely connected with Fourier series, they often appear when differential or integral equations are discretized, they arise in physical data-processing applications, in

the theories of orthogonal polynomials, stationary processes, and moment problems; see Heinig and Rost [31]. For other references on Toeplitz matrices see [26], [28] and [33].

A special case of a Toeplitz matrix is a circulant matrix, where each row is rotated one element to the right relative to the preceding row. Circulant matrices and their properties have been studied in [14] and [28]. In numerical analysis circulant matrices are important because they are diagonalized by a discrete Fourier transform, and hence linear equations that contain them may be quickly solved using a fast Fourier transform. These matrices are also very useful in digital image processing.

A directed or undirected graph whose adjacency matrix is circulant is called *circulant*. Circulant graphs and their properties such as connectivity, hamiltonicity, bipartiteness, planarity and colourability have been studied by several authors (see [8], [11], [15], [25], [35], [38], [41] and [24]). In particular, the conjecture of Boesch and Tindell [8], that all undirected connected circulant graphs are hamiltonian, was proved by Burkard and Sandholzer [11].

A directed or undirected *Toeplitz graph* is defined by a Toeplitz adjacency matrix. The properties of Toeplitz graphs; such as bipartiteness, planarity and colourability, have been studied in [18], [19], [20]. Hamiltonian properties of undirected Toeplitz graphs have been studied in [13] and [32].

For arbitrary digraphs the hamiltonian path and cycle problems are also very difficult and both are $\mathcal{NP}$-complete (see, e.g. the book [22] by Garey and Johnson). It is worthwhile mentioning that the hamiltonian cycle and path problems are $\mathcal{NP}$-complete even for some special classes of digraphs. Garey, Johnson and Tarjan shows [23] that the problem remains $\mathcal{NP}$-complete even for planar 3-regular digraphs. Some powerful necessary conditions, due to Gutin and Yeo [10], are considered for a digraph to be hamiltonian. For information on hamiltonian and traceable digraphs, see e.g. the survey [2] and [3] by Bang-Jensen and Gutin, [9] by Bondy, [29] by Gutin and [39] by Volkmann.

In this thesis, we investigate the hamiltonicity of directed Toeplitz graphs. The main purpose of this thesis is to offer sufficient conditions for the existence of hamiltonian paths and cycles in directed Toeplitz graphs, which we will discuss in Chapters 3 and 4.

The main diagonal of an $(n \times n)$ Toeplitz adjacency matrix will be labeled 0 and it contains only zeros. The $n-1$ distinct diagonals above the main diagonal will be labeled $1, 2, \ldots, n-1$ and those under the main diagonal will also be labeled $1, 2, \ldots, n-1$. Let $s_1, s_2, \ldots, s_k$ be the upper diagonals containing ones and $t_1, t_2, \ldots, t_l$ be the lower diagonals containing ones, such that $0 < s_1 < s_2 < \cdots < s_k < n$ and $0 < t_1 < t_2 < \cdots < t_l < n$. Then, the corresponding directed Toeplitz graph will be denoted by $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$. That is, $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$ is the graph with vertex set $1, 2, \ldots, n$, in which the edge (i, j), $1 \leq i < j \leq n$, occurs if and only if $j - i = s_p$ or $i - j = t_q$ for some p and q $(1 \leq p \leq k, 1 \leq q \leq l)$.

In Chapter 1 we describe some basic ideas, terminology and results about graphs and digraphs. Further we discuss adjacency matrices, Toeplitz matrices, which we will encounter in the following chapters.

In Chapter 2 we discuss hamiltonian graphs and add a brief historical note. We then discuss undirected Toeplitz graph, and finally mention some known results on hamiltonicity of undirected Toeplitz graphs found by van Dal et al. [13] and C. Heuberger [32].

Since all graphs in the main part of the thesis (Chapters 3 and 4) will be directed, we shall omit mentioning it in these chapters. We shall consider here just graphs without loops, because loops play no role in hamiltonicity investigations. Thus, unless otherwise mentioned, in Chapters 3 and 4, by a graph we always mean a finite simple digraph.

In Chapter 3, for $k = l = 1$ we obtain a characterization of cycles among directed Toeplitz graphs, and another result similar to Theorem 10 in [13]. Directed Toeplitz graphs with $s_1 = 1$ (or $t_1 = 1$) are obviously traceable. If we ask moreover that $s_2 = 2$, we see that the hamiltonicity of $T_n\langle 1, 2; t_1\rangle$ depends upon the parity of t_1 and n. Further in the same Chapter, we require $s_3 = 3$ and succeed to prove the hamiltonicity of $T_n\langle 1, 2, 3; t_1\rangle$ for all t_1 and n.

In Chapter 4 we present a few results on Toeplitz graphs with $s_1 = t_1 = 1$ and $s_2 = 3$. They will often depend upon the parity of n.

Chapter 5 contains some concluding remarks.