Abstract

Mathematical inequalities play very important role in development of all branches of mathematics. A huge effort has been made to discover the new types of inequalities and the basic work published in 1934 by Hardy, Littlewood and Pólya [36]. Later on Beckenbach and Bellman in 1961 in their book "Inequalities"[13], and the book "Analytic inequalities"by Mitronović [53] published in 1970 made considerable contribution in this field. The mathematical inequalities are useful because these are used as major tool in the development of modern analysis. A wide range of problems in various branches of mathematics are studied by well known Jensen, Hilbert, Hadamard, Hardy, Poincáre, Opial, Sobolev, Levin and Lyapunov inequalities. In 1992, J. Pečarić, F. Proschan and Y. L. Tong play their vital role in this field and they published famous book "Convex Functions, Partial Orderings and Statistical Application"which is considered as a brightening star in this field.

On the other hand, the applications of fractional calculus in mathematical inequalities have great importance. Hardy-type inequalities are very famous and play fundamental role in mathematical inequalities. Many mathematicians gave generalizations, improvements and application in the development of the Hardy's inequalities and they use fractional integrals and fractional derivatives to establish new integral inequalities. Further details concerning the rich history of the integral inequalities can be found in [58]–[64], [73]–[75] and the references given therein.

Čižmešija, Krulić, Pečarić and Persson establish some new refined Hardy-type inequalities with kernels in their recent papers [4], [25], [28], [29], [34], [52] (also see

[15] [23]). Inequalities lies in the heart of the mathematical analysis and numerous mathematicians are attracted by these famous Hardy-type inequalities and discover new inequalities with kernels and applications of different fractional integrals and fractional derivatives, (see [25], [28], [38], [50], [52], [65]).

In this Ph.D thesis an integral operator with general non-negative kernel on measure spaces with positive σ-finite measure is considered. Our aim is to give the inequality of G. H. Hardy and its improvements for Riemann-Liouville fractional integrals, Canavati-type fractional derivative, Caputo fractional derivative, fractional integral of a function with respect to an increasing function, Hadamard-type fractional integrals and Erdélyi-Kober fractional integrals with respect to the convex and superquadratic functions. We will use different weights in this construction to obtain new inequalities of G. H. Hardy. Such type of results are widely discussed in [38](see also [28]). Also, we generalize and refine some inequalities of classical Hardy-Hilbert-type, classical Hardy-Littlewood-Pólya-type and Godunova-type inequalities [55] for monotone convex function.

The first chapter contains the basic concepts and notions from theory of convex functions and superquadratic functions. Some useful lemmas related to fractional integrals and fractional derivatives are given which we frequently use in next chapters to prove our results.

In the second chapter, we state, prove and discuss new general inequality for convex and increasing functions. Continuing the extension of our general result, we obtain new results involving different fractional integrals and fractional derivatives. We give improvements of an inequality of G. H. Hardy for convex and superquadratic functions as well.

In the third chapter, we give the new class of the G. H. Hardy-type integral inequalities with applications. We provide some generalized G. H. Hardy-type inequalities for fractional integrals and fractional derivatives.

In fourth chapter, we present generalized Hardy's and related inequalities involving monotone convex function. We generalize and refine some inequalities of classical

Pólya-Knopp's, Hardy-Hilbert, classical Hardy-Littlewood-Pólya, Hardy-Hilber-type and Godunova's. We also give some new fractional inequalities as refinements.

In the fifth chapter, we establish a generalization of the inequality introduced by D. S. Mitrinović and J. Pečarić in 1988. We prove mean value theorems of Cauchy type and discuss the exponential convexity, logarithmic convexity and monotonicity of the means. Also, we produce the n-exponential convexity of the linear functionals obtained by taking the non-negative difference of Hardy-type inequalities. At the end, some related examples are given.