

## Abstract

In the modern day language, finding quasi Banach operator ideals admitting eigenvalue type  $\ell_s$  for  $0 < s < \infty$  and those admitting some traces -for example the spectral trace, were two important problems set in 20th century. We set out with a description of eigenvalue type of  $\mathfrak{N}_r$  for  $0 < r \leq 1$  restricted to subspaces of  $L_p$ -spaces and the coincidence of nuclear and spectral traces on them. We proceed and consider operator ideals  $\mathfrak{N}_{r,p}$  and  $\mathfrak{N}^{r,p}$  -where  $p$  and  $r$  are related, and describe its eigenvalue type with many different techniques -all of which are important in themselves. We observe strong relationships among eigenvalue type, nuclear and spectral traces and approximation properties of type  $(r, p)$ ;  $AP_{r,p}$  for short. We prove that if  $0 < s \leq 1$  and  $1/r = 1/s + |1/p - 1/2|$ , then every Banach space  $X$  has  $AP_{r,p}$  ( and also  $AP^{r,p}$ ). The optimality of our results for the eigenvalue types is provided to show that there is no hope for any improvements in it. A negative answer to a question of A. Pietsch and A. Hinrichs is provided; they asked for the validity of the inclusion  $\mathfrak{N}_r^d \subset \mathfrak{N}_r$ , where  $0 < r < 1$ . Independently and without knowing that it was considered by E. Oja et al., we have replied with a negative answer to the open question posed by two Indian mathematicians Anil K. Karn and D. P. Sinha. They didn't know if there existed a Banach space without approximation property of type  $p$  for  $1 \leq p < 2$ .