Abstract

Studies on the existence and properties of paths and cycles through a specified number of vertices in a graph have been of considerable interest to both pure and applied mathematicians as well as researchers in other disciplines. Applications of such studies can be found in fields related to electrical engineering, computer algorithm analysis, operations research and other areas of scientific research.

Since the four-colour theorem has been proven finally with the help of a computer, the oldest and the most famous unsolved problem in the theory of graphs is undoubtedly that of finding an elegant and practical characterization of hamiltonian graphs. Indeed, these two problems are not entirely unrelated. It is known that [40] every hamiltonian plane map is 4-colourable. The problem of recognising a graph to be hamiltonian is notoriously difficult. In fact, Karp [29] proved that it is an $\mathcal{NP}$-complete problem. Combined with a theorem of S.A Cook [10], the existence of a good characterization of nonhamiltonian graph seems unlikely, although several necessary conditions and many sufficient conditions (see [3]) have been discovered. The interested reader is referred in particular to the surveys of Berge ([2], Chapter 10), Bondy and Murty ([6], Chapters 4 and 9), J. C. Bermond [3], Flandrin, Faudree and Ryjáček [20] and R. Gould [24].

A *Toeplitz matrix*, is a square matrix $(n \times n)$ which has constant values along all diagonals parallel to the main diagonal. Toeplitz matrices have uses in different areas in pure and applied mathematics. For example, they are closely connected with Fourier series, they often appear when differential or integral equations are discretized, they arise in physical data-processing applications, in the theories of orthogonal polynomials, stationary processes, and moment problems; see Heinig and Rost [26]. For other references on Toeplitz matrices see [25], [23] and [28].

A special case of a Toeplitz matrix is a circulant matrix, where each row is rotated one element to the right relative to the preceding row. Circulant matrices and their properties have been studied in [13] and [25]. In numerical analysis circulant matrices

are important because they are diagonalized by a discrete Fourier transform, and hence linear equations that contain them may be quickly solved using a fast Fourier transform. These matrices are also very useful in digital image processing.

A graph whose adjacency matrix is circulant is called *circulant*. Circulant graphs and their properties such as connectivity, hamiltonicity, bipartiteness, planarity and colourability have been studied by several authors (see [5], [7], [21], [22] and [41]). In particular, the conjecture of Boesch and Tindell [5], that all undirected connected circulant graphs are hamiltonian, was proved by Burkard and Sandholzer [7].

Hamiltonicity in many classes of graphs has been studied so far. One special class of graphs whose hamiltonicity has been studied is that of Toeplitz graphs, introduced by R. van Dal, G. Tijssen, Z. Tuza, J.A.A. van der Veen, Ch. Zamfirescu and T. Zamfirescu [12] in 1996. C. Heuberger [27] continued the study of hamiltonicity of Toeplitz graphs in 2002. Hamiltonian properties of directed Toeplitz graphs have been investigated in [32], [34], and [33]. Moreover, hamiltonian connectedness of directed Toeplits graphs was studied in [35].

A simple graph T of order n is called a *Toeplitz graph* if its adjacency matrix $A(T)$ is Toeplitz. Some other properties of Toeplitz graphs; such as bipartiteness, planarity and colourability, have been studied in [17], [18] and [19].

In this thesis, we investigate the hamiltonian connectedness of undirected Toeplitz graphs.

In Chapter 1 we describe some basic definitions, and results about graphs. We also discuss adjacency matrices, Toeplitz matrices and Toeplitz graphs.

Chapter 2 deals with a brief history of the hamiltonian graphs. We then mention some known results on the hamiltonicity of Toeplitz graphs.

In Chapter 3, we present results about the hamiltonian connectedness of Toeplitz graphs, more precisely of $T_n\langle t_1, t_2\rangle$, $T_n\langle 1, t, s\rangle$.

In Chapter 4 we improve our results for the case of $T_n\langle 1, 3, s\rangle$. and $T_n\langle 1, 4, s\rangle$.