Abstract

Steffensen [28] proved the following inequality: Assume that two integrable functions f and g are defined on interval $[a,b]$ such that f is non-increasing and that $0 \le g(t) \le 1$ on $[a,b]$. Then

$$\int_{b-\gamma}^{b} f(t)\,dt \le \int_{a}^{b} f(t)g(t)\,dt \le \int_{a}^{a+\gamma} f(t)\,dt. \tag{1}$$

where

$$\gamma = \int_a^b g(t)\,dt \tag{2}$$

Steffensen's inequality resurfaced in another paper [27] of his, 29 years after its first publication and it took another 10 years or so before it caught Bellman's attention in [4]. Since then, a few hundred papers have been devoted to various aspects of generalizations of it. See for instance Mitrinović [18] or, more recently, Pečarić see [22] and references therein. Moreover, Hayashi [13], generalized inequality (1) slightly by imposing the condition: $0 \le g(t) \le A$ where A is any positive real number. Davies [18] proved Steffensen's inequality by using another technique. However, in 1984, Vasić and Pečarić proved that Davies's work follows from Steffensen's inequality as well. Vasić and Pečarić proved proved result (see [29] also [22]) which gives necessary and sufficient conditions on g in (1). In addition, Pečarić gave other generalizations of (1) and its converse see [21], [24] and [25]. Rabier's, in [26], also proved generalization of Steffensen's inequality. Rabier's generalization of (1) has connection with the work of Pečarić [21]. Moreover, Rabier also proved generalized Steffensen-type inequalities for higher dimensions.

In this dissertation, most of our work surrounds around generalization of (1) with different techniques. As a result, many interesting inequalities have been obtained. This thesis is structured as follows:

In the first chapter we recalled notions related to convex functions and generalized convex functions. In addition, Steffensen's inequality, generalizations of Steffensen's inequality and recent work and its connection with Steffensen's inequality has also been given. In the second chapter, we obtained inequalities which are improvements and conversions of inequalities in [26]. These results enable us to generate new class of exponentially convex functions and construct means of Gini-Dresher type. As an application, we give improvement of inequality which compares integrals $\int_{\mathbb{R}^N} |f(x)||x|^{\beta}\,dx$ and $\int_{\mathbb{R}^N} |f(x)|\,dx = \|f\|_1$. In the next chapter, generalization

of Steffensen's inequality by extending the results of Pečarić [21] and Rabier [26] has been given. Moreover, by using the obtained results we gave Hardy-type inequality, its improvement and family of exponentially convex functions.

In next two chapters, we give Lidstone and Hermite interpolation of composition function and some related inequalities. The obtained results are closely related to a generalization of Steffensen's inequality given in [10]. By using the obtained results, we generate new families of exponentially convex functions. In the final chapter, another generalization of Steffensen's inequality by generalizing the results from [21] has been given. It is obtained by applying Abel-Gontscharoff interpolation to composition functions. These inequalities enable us to produce new class of exponentially convex function.