ABSTRACT

The concept of semirings was introduced by H.S. Vandiver in 1934 [73]. A semiring is a non-empty set S with operations "+" and "." such that $(S,+)$ is a commutative monoid with an absorbing zero 0 i.e. $u0 = 0 = 0u$, $u+0 = u = 0+u\ \forall u \in S$, $(S,.)$ is a semigroup and "." is left as well as right distributive over "+" i.e. $a(b+c) = ab+ac$ and $(b+c)a = ba+ca\ \forall a,b,c \in S$. In this thesis, the basic environment in which we will work is the domain of semirings and fundamental references of the theory of semirings are [40–43]. Semirings are natural generalizations of rings and distributive lattices. They are found frequently in formal languages, theory of automata, and many other branches of applied and pure mathematics [43]. Semirings have many applications in Artificial intelligence, Computer programming [19, 43], Optimization theory, Graph theory, and Dynamical systems [42, 43]. In recent years, the Fuzzy theory of semirings becomes an interesting field for researchers (see [4] for ready reference). Due to its enormous applications and natural implications, many authors have attempted to extend its different aspects. P.H. Karvellas [55] enhanced the essence of its analysis by introducing pseudo-inverses in this structure. He termed S as an additively inversive semiring (or inverse semiring) if for each element u in S, there exists a unique element $u' \in S$ such that $u + u' + u = u$ and $u' + u + u' = u'$, where element u' is called pseudo-inverse of u. This extension paved new paths to extend some important results of algebra in this generalized algebraic structure. However, due to a lack of notion of commutators and undefinable Jacobi identities, the Lie theory of semirings was not developed until a novel concept of MA-semirings by M.A. Javed, M. Aslam, and M. Hussain [50]. They attached (A2)-condition of Bandlet and Petrich [10] to additive inverse semirings i.e. $a + a' \in Z(S)\ \forall a \in S$, where $Z(S)$ is the centre of S and named it as MA-semirings. This notion sets the stage for researchers to use commutators and develop Lie type theory in semirings [51, 52, 68, 69]. One of the important areas of ring theory is the study of functional identities. The main goal of the study of functional identities is to determine the form of mappings satisfying certain functional relations on different subsets of rings involved or, if this is not possible, to determine the structure of the underlined rings [27]. The theory of functional identities has many applications in Linear algebra, Functional analysis, Operator theory, and Mathematical physics [25–27]. The theory of MA-semiring has turned out to be a powerful tool for generalizing many results of functional identities of rings in MA-semirings. In this thesis, we generalize some important results of the theory of rings in MA-semirings.

This thesis is divided into eight chapters.

In the first chapter, we describe briefly some basic notions and facts related to rings and semirings to make this thesis self-contained.

In chapter 2, we study a generalized form of centralizers on MA-semiring. Our focus is to investigate conditions that turn an additive mapping into a generalized centralizer. The results of this chapter have been published in "Journal of Mechanics of Continua and Mathematical Sciences" [2].

In chapter 3, we extend the study of derivations of rings to MA-semirings and find some conditions on derivations that make MA-semirings commutative. We also introduce a concept of proper MA-semirings and study some functional identities in this structure which are otherwise trivial in rings. In the last section of this chapter, we use the notion of centralizers to investigate the commutativity conditions of MA-semirings. Some results of this chapter have been published in "Journal of Mechanics of Continua and Mathematical Sciences" [64].

In chapter 4, we generalize some important results of rings of M. Brešar [24] in semirings by probing conditions that make the skew-commuting map zero. Some material from this chapter is published in the journal "Quasigroups and Related Systems" [62].

In chapter 5, we study n-skew-commuting mappings and prove a conjecture raised by A. Fošner and N. Rehman [37] which is stated as:
Let X be a (2n)!-torsion free semiprime ring and $f : X \to X$ be an additive map, then $f(u)u^n + u^n f(u) = 0 \, \forall u \in X$ implies $f = 0$. Furthermore, we generalize this result in the setting of MA-semirings. We also study an Engel condition for an additive map in rings that converts an additive map into a commuting map. Some results of this chapter are a part of our publication in the journal "General Mathematics Notes" [61].

In chapter 6, we introduce notions of skew-Engel conditions and mixed-Engel conditions in rings as well as in MA-semirings. We also investigate the behavior of the additive mapping on said structures. The material of this chapter has been published in "Journal of Algebra and Related Topics" [63].

In chapter 7, we study some skew-commuting traces of multi-additive maps in rings. We also extend these results in the setting of MA-semiring.

In the last chapter, we initiate a study of generalized m-derivations and extend a well-known Posner's theorem [72] on derivations to generalized m-derivations. The material of this chapter is published in the journal "International Journal of Grid and Distributed Computing" [3].

We conclude this thesis with some future work related to our research.