Abstract

The subject of this thesis is the study of some properties of some ideals associated with the Proper Interval (PI) graphs.

In Chapter 1, some necessary definitions and results from combinatorics and commutative algebra are presented.

In Chapter 2, we prove that the Betti number conjecture is true for the PI graph G with $\operatorname{reg}(S/J_G) = 2$.

Chapter 3 is a contribution to the study of Koszulness of binomial edge ideals of pairs of graphs. The main result is Theorem 1.1.31 which states that, for two connected graphs G_1, G_2 on the vertex sets $[m], [n]$ respectively, with $m, n \geq 3$, the following statements are equivalent:

(i) The pair of graphs (G_1, G_2) is Koszul.

(ii) G_1 is a PI graph and G_2 is complete or vice-versa.

(iii) J_{G_1,G_2} has a quadratic Gröbner basis with respect to the lexicographic order, $<_{\text{lex}}$, induced by $x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots > x_{2n} > \cdots > x_{m1} > \cdots > x_{mn}$.

(iv) J_{G_1,G_2} has a quadratic Gröbner basis with respect to the revlexicographic order, $<_{\text{rev}}$, induced by $x_{1n} > x_{2n} > \cdots > x_{mn} > x_{1n-1} > \cdots > x_{mn-1} > \cdots > x_{11} > \cdots > x_{m1}$.

(v) The graded maximal ideal of $R = K[x_{11}, \ldots, x_{mn}]/J_{G_1,G_2}$ has linear quotients with respect to the following order of its generators:

$$\overline{x}_{m1}, \overline{x}_{m-1,1}, \ldots, \overline{x}_{11}, \overline{x}_{m2}, \ldots, \overline{x}_{12}, \ldots, \overline{x}_{mn}, \ldots, \overline{x}_{1n}.$$

For the proof we use a pure combinatorial characterization of PI graphs for which we provide a new proof.