Abstract

In this work we study ideals generated by 2-minors of Hankel matrices which arise from combinatorics.

Let \( G \) be a closed graph, that is, a chordal graph on the vertex set \([n]\) with the maximal cliques \( F_1 = [a_1, b_1], \ldots, F_r = [a_r, b_r] \) where \( 1 = a_1 < a_2 < \cdots < a_r < b_r = n \), and
\[
\begin{pmatrix}
x_1 & \cdots & x_{n-1} & x_n \\
x_2 & \cdots & x_n & x_{n+1}
\end{pmatrix}
\]
the generic \( 2 \times n \)-Hankel matrix.

We define the ideal \( I_G \subset K[x_1, \ldots, x_n, x_{n+1}] \) generated by the 2-minors of this matrix which correspond to the edges of \( G \). More precisely,
\[
I_G = (x_i x_{j+1} - x_j x_{i+1} : i < j \text{ and } \{i, j\} \in E(G)).
\]

The ideal \( I_G \) is called the Hankel ideal associated with the graph \( G \). In Chapter 2, we study algebraic and homological properties of \( I_G \).

In Chapter 3, we extend the construction of the previous chapter to arbitrary generic Hankel matrices. Let \( m, n \geq 3 \) be two integers and \( X \) the generic \( m \times n \)-Hankel matrix. We consider \( G_1 \) and \( G_2 \) closed graphs on the vertex set \([m]\) and, respectively, \([n]\).

We define the Hankel ideal associated with the pair \( (G_1, G_2) \) as the ideal generated by the 2-minors of \( X \) of the form \( g_{e,f} = [ij|k\ell] \) where \( i < j, k < \ell \) and \( \{i, j\} \in E(G_1), \{k, \ell\} \in E(G_2) \). We prove that \( I_{(G_1, G_2)} \) coincides with the Hankel ideal \( I_G \) associated to an appropriate closed graph \( G \) on the vertex set \([m + n - 2]\). Therefore, we may apply all the results of Chapter 2 in order to derive the properties of \( I_{(G_1, G_2)} \) in terms of the combinatorial data of the graphs \( G_1, G_2 \).